# Uniqueness theorems for equations of Keldysh Type

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**Abstract**

A fundamental result that characterizes elliptic-hyperbolic equations of Tricomi type, the uniqueness of classical solutions to the open Dirichlet problem, is extended to a large class of elliptic-hyperbolic equations of Keldysh type. The result implies the non-existence of classical solutions to the closed Dirichlet problem for this class of equations. A uniqueness theorem is also proven for a mixed Dirichlet-Neumann problem. A generalized uniqueness theorem for the adjoint operator leads to the existence of distribution solutions to the closed Dirichlet problem in a special case. *MSC2000*: 35M10

## 1 Introduction

In 1956, Morawetz [13] proved the uniqueness of smooth solutions to *open* Dirichlet problems, having data prescribed on only part of the boundary, for certain mixed elliptic-hyperbolic equations of Tricomi type. That result implied that the *closed* Dirichlet problem, in which data are prescribed on the entire boundary, is over-determined for smooth solutions of such equations.

Morawetz's result was later extended to a large class of boundary value problems for Tricomi-type equations, by Manwell ([11] and Sec. 16 of [12]) and by Morawetz herself [15]. But there is as yet no corresponding result for equations of Keldysh type – the other canonical local form for second-order linear equations of mixed elliptic-hyperbolic type. Morawetz's result has been extended to a particular equation which is of Keldysh type in any neighborhood of the origin and of Tricomi type elsewhere [16]. That proof

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requires the type-change function to be symmetric about the $x$-axis and an analytic function of its arguments, neither of which we assume. In Sec. 2 we generalize the assertion of [16] to an entire class of equations under mild restrictions on the type-change function. While that result is by no means unexpected, its apparent absence from the literature until now, nearly 80 years after the derivation by Cinquini-Cibrario of the two canonical forms for elliptic-hyperbolic equations [4], is rather surprising.

Morawetz considered equations essentially having the form

$$\mathcal{K}(y)u_{xx} + u_{yy} = 0, \tag{1.1}$$

where

$$\mathcal{K}(0) = 0 \tag{1.2}$$

and

$$y\mathcal{K}(y) > 0 \text{ for } y \neq 0. \tag{1.3}$$

Here and below, the unknown function $u$ depends on $(x, y) \in \mathbb{R}^2$. Equations having the form (1.1), possibly including lower-order terms and satisfying conditions (1.2) and (1.3), are said to be of *Tricomi type*. Equations having the form

$$\mathcal{K}(x)u_{xx} + u_{yy} + \text{ lower-order terms } = 0, \tag{1.4}$$

where $\mathcal{K}(x)$ satisfies (1.2) and

$$x\mathcal{K}(x) > 0 \text{ for } x \neq 0, \tag{1.5}$$

are said to be of *Keldysh type*.

In Sec. 2 we consider a special case of these equations, namely, equations having the form

$$Lu \equiv \mathcal{K}(x)u_{xx} + u_{yy} + \frac{\mathcal{K}'(x)}{2}u_x = 0, \tag{1.6}$$

where $\mathcal{K}$ satisfies conditions (1.2) and (1.5). We assume for convenience that $\mathcal{K}$ is $C^1$, and monotonic on the hyperbolic region, but the result clearly extends to weaker hypotheses on $\mathcal{K}$. For example, the monotonicity hypothesis on $\mathcal{K}$ is imposed only in order to simplify the graphs of the characteristic lines, which in turn simplifies the proof of Theorem 2.1 in the hyperbolic region. See also the discussion of eq. (2.10), below.

As an example, choose $\mathcal{K}(x) = x^{2k_0-1}$ for $k_0 \in \mathbb{Z}^+$. The operator $L$ under this choice of type-change function is roughly analogous, for equations of Keldysh type, to the well known *Gellerstadt operator* [6] for equations of

Tricomi type. Other examples are the polar-coordinate forms of the equation for harmonic fields on the extended projected disc (eq. (16) of [18]) and an equation arising from a uniform asymptotic approximation of high-frequency waves near a caustic (eq. (4.1) of [10]). Both equations can be put into the form (1.6) in the cartesian $r\theta$-plane, with the $x$-axis replaced by the line $r = 1$.

The main object of Sec. 2 is to show that any smooth solution of eq. (1.6) which vanishes identically on the non-characteristic boundary of a typical domain also vanishes in the interior (Theorem 2.1). This implies the nonexistence of classical solutions to a closed Dirichlet problem (Corollary 2.2). We also show that an analogous result holds if only the normal derivative of the solution vanishes on the horizontal arcs of the non-characteristic boundary (Theorem 2.3).

In the important special cases in which $\mathcal{K}(x) = x$ or $L$ is formally self-adjoint, we prove uniqueness theorems (in a very generalized sense) for solutions to the adjoint equation. This allows us to show the existence of distribution solutions to a closed Dirichlet problem by so-called *projection* methods (Theorems 3.3 and 3.4). Given these results, it is natural to wonder about the existence of weak solutions to closed boundary value problems. This is a difficult question. Some obstacles to proving the existence of weak solutions to a closed Dirichlet problem by the direct application of even powerful projection methods are discussed in Sec. 5.1 of [17]. However, the existence of weak solutions to the closed Dirichlet problem for the equation studied in [16] has been proven [19].

## 2 The nonexistence of classical solutions

For given $\mathcal{K}(x)$, define constants $a$, $b$, $d$, and $m$, where $m < a \le 0 < d$ and $b > 0$. Consider the domain $\mathcal{D}$ formed by the line segments

$$\mathcal{L}_1 = \{(x,y)\,|a \le x \le d, y = -b\}\,;$$

$$\mathcal{L}_2 = \{(x,y)\,|x = d, -b \le y \le b\}\,;$$

$$\mathcal{L}_3 = \{(x,y)\,|a \le x \le d, y = b\}\,;$$

the characteristic line $\Gamma_1$ joining the points $(m,0)$ and $(a,-b)$; and the characteristic line $\Gamma_2$ joining the points $(m,0)$ and $(a,b)$.

The $y$-axis divides the domain $\mathcal{D}$ of eq. (1.6) into the subdomains

$$\mathcal{D}^+ = \{(x,y) \in \mathcal{D} | x \ge 0\}\,,$$

and $\mathcal{D}^- = \mathcal{D} \backslash \mathcal{D}^+$. Equation (1.6) is (non-uniformly) elliptic for $(x,y) \in \mathcal{D}^+$.

**Theorem 2.1.** *Let $u(x,y)$ be a twice-differentiable solution of eq. (1.6), with $\mathcal{K}$ satisfying conditions (1.2) and (1.5). Assume that $\mathcal{K}$ is $C^1$, and monotonic on $\mathcal{D}^-$. If $u$ vanishes on the non-characteristic boundary, then $u \equiv 0$ on all of $\mathcal{D}$.*

*Proof.* We follow the approach of [13], [15], and [16]. Introducing the auxiliary function

$$I = \int_0^{(x,y)} \left[\mathcal{K}(x)u_x^2 - u_y^2\right] dy - 2u_x u_y dx,$$

we compute

$$\begin{aligned} &\frac{\partial}{\partial y}(-2u_x u_y) - \frac{\partial}{\partial x}\left[\mathcal{K}(x)u_x^2 - u_y^2\right] \\ &= (-2u_{xy}u_y - 2u_x u_{yy}) - \left[\mathcal{K}'(x)u_x^2 + 2\mathcal{K}(x)u_x u_{xx} - 2u_y u_{yx}\right] \\ &= -2u_x u_{yy} - \left[\mathcal{K}'(x)u_x^2 + 2\mathcal{K}(x)u_x u_{xx}\right] \\ &= -2u_x\left(u_{yy} + \frac{\mathcal{K}'(x)}{2}u_x + \mathcal{K}(x)u_{xx}\right) = 0, \end{aligned}$$

using (1.6) and the equivalence of mixed partial derivatives. We conclude that there exists a function $\xi(x,y)$ such that

$$\xi_x = -2u_x u_y \tag{2.1}$$

and

$$\xi_y = \mathcal{K}(x)u_x^2 - u_y^2. \tag{2.2}$$

We will first show that $u$ vanishes identically in $\mathcal{D}^+$. To accomplish this, we must show that $u$ vanishes identically on the sonic line $x = 0$. Once we have shown that, we will have zero boundary conditions on $\mathcal{D}^+$. We will complete the proof for the elliptic region by invoking a maximum principle for non-uniformly elliptic equations.

Because $u \equiv 0$ on $\mathcal{L}_1$, we conclude that $u_x$ vanishes identically on that horizontal line. Thus we have, by (2.1),

$$\xi_x = 0 \text{ on } \mathcal{L}_1. \tag{2.3}$$

Also, $u \equiv 0$ on $\mathcal{L}_3$, so $u_x = 0$ on that horizontal line as well, implying that

$$\xi_x = 0 \text{ on } \mathcal{L}_3. \tag{2.4}$$

Equations (2.3) and (2.4) imply that, on $\mathcal{L}_1$ and $\mathcal{L}_3$, $\xi$ is a function of $y$ only. But $y$ is constant on those two horizontal lines, implying that

$$\xi = c_1 \text{ on } \mathcal{L}_1$$

and

$$\xi = c_2 \text{ on } \mathcal{L}_3,$$

where $c_1$ and $c_2$ are constants. On the line $\mathcal{L}_2$, $u \equiv 0$, implying that $u_y = 0$ on that vertical line. Also, $\mathcal{K}(x) > 0$ on $\mathcal{L}_2$. These facts imply, using eq. (2.2), that $\xi_y \geq 0$ on $\mathcal{L}_2$, which in turn implies that

$$c_2 \geq c_1. \tag{2.5}$$

On the line $x = 0$, $\mathcal{K} = 0$, implying by (2.2) that $\xi_y \leq 0$ on that vertical line. This is turn implies that

$$c_2 \leq c_1. \tag{2.6}$$

Inequalities (2.5) and (2.6) are in contradiction unless $c_1 = c_2$. Taking into account that $\xi$ cannot increase with increasing $y$ on the line $x = 0$, it also cannot decrease with increasing $y$, as it would then have to increase in order to return to its initial value at the endpoint. This implies that $\xi_y = 0$ on the $y$-axis. Using (2.2) again, we find that on the $y$-axis,

$$-\,u_y^2 = 0, \tag{2.7}$$

so the function $u\,(0, y)$ is constant there. Because

$$u\,(0, -b) = u\,(0, b) = 0,$$

that constant is zero. Thus on the rectangle $\partial\mathcal{D}^+$ we have a closed Dirichlet problem having homogeneous boundary conditions.

A well known extension of the maximum principle to non-uniformly elliptic operators (Proposition A.1) implies that the smooth function $u$ attains both its maximum and minimum values on the boundary. Because it is identically zero there, $u$ must be zero in all of $\mathcal{D}^+$.

We obtain the identical vanishing of $u$ on the hyperbolic region by integration along characteristic lines as in [1]. We have

$$d\xi = \xi_x dx + \xi_y dy = (-2u_x u_y)\,dx + \left[\mathcal{K}(x)u_x^2 - u_y^2\right] dy.$$

On characteristic lines,

$$dx = \pm\sqrt{-\mathcal{K}(x)}dy$$

and

$$d\xi = \left[\mp 2u_x u_y \sqrt{-\mathcal{K}(x)} + \mathcal{K}(x)u_x^2 - u_y^2\right] dy$$
$$= -\left[\sqrt{-\mathcal{K}(x)}u_x \pm u_y\right]^2 dy \leq 0. \tag{2.8}$$

Thus $\xi$ is non-increasing in $y$ on any arbitrarily chosen characteristic.

Initially, take $a = 0$.

Because $u \equiv 0$ on the sonic line $x = 0$, we conclude that $u_y = 0$ on that vertical line. So $\xi_x = 0$ on the sonic line by (2.1). Because $K(0) = 0$, we conclude that $\xi_y = 0$ on the sonic line by (2.2) and (2.7). Beginning at the point $(0, -b)$, proceed along $\Gamma_1$ to $(m, 0)$ and then along $\Gamma_2$ to $(0, b)$. Expression (2.8) implies that $\xi$ will not increase in $y$ along this path from $(0, -b)$ and $(0, b)$. Because $\xi$ is equal to the same constant at those two points, $\xi$ must be constant in $y$ along $\Gamma_1 \cup \Gamma_2$. (If $\xi$ decreased in $y$ at any point along such its path, it would have to increase in $y$ at a later point in order to return to its constant value at $(0, b)$. And it cannot increase in $y$ along a characteristic.) Ascending along the $y$-axis from the point $(0, -b)$, for any initial point above $(0, -b)$ and any terminal point below $(0, b)$ on the $y$-axis we can always find a pair of characteristic lines intersecting at some point on the $x$-axis to the right of $(m, 0)$. We conclude that $\xi_y = 0$ on $\mathcal{D}^-$. But then (2.2) implies that

$$\mathcal{K}(x)u_x^2 = u_y^2 \text{ on } \mathcal{D}^-. \tag{2.9}$$

Because $\mathcal{K} < 0$ on $\mathcal{D}^-$, we are forced to conclude from (2.9) that $u_x = u_y = 0$ on $\mathcal{D}^-$. This is turn implies that $u$ is constant on $\mathcal{D}^-$. Because $u \equiv 0$ on the sonic line, that constant must be zero by the smoothness of $u$.

Now take $a < 0$. Because $\xi_x = 0$ on $\mathcal{L}_1$ and $\mathcal{L}_3$, $\xi$ remains constant between $(0, -b)$ and $(a, -b)$ and between $(0, b)$ and $(a, b)$. Moreover, $\xi_y$ remains non-positive along $\Gamma_1$ and $\Gamma_2$. As we move the initial and terminal points to the right along $\mathcal{L}_1$ and $\mathcal{L}_3$ in $\mathcal{D}^-$, we can always find a pair of characteristic lines which intersect at a point on the $x$-axis to the right of $(m, 0)$. Arguing as in the case $a = 0$, we again conclude that $u \equiv 0$ on $\mathcal{D}^-$. This completes the proof of Theorem 2.1. $\square$

In the special case in which $\mathcal{K}$ is an analytic function, we do not require a maximum principle, so we do not need to show that $u = 0$ on the line $x = 0$. Rather, we observe that $u_y = 0$ on $\mathcal{L}_2$ as $u \equiv 0$ on that vertical line. Our analysis of the constants $c_1$ and $c_2$ implies that $\xi_y = 0$ on $\mathcal{L}_2$ as well. Because in addition, $\mathcal{K} > 0$ on $\mathcal{L}_2$, eq. (2.2) implies that $u_x = 0$ on $\mathcal{L}_2$. We use this

last identity as Cauchy data for the Cauchy-Kowalevsky Theorem, to argue that $u$ remains equal to zero as one moves in the negative $x$-direction away from $\mathcal{L}_2$ along the rectangle $\mathcal{D}^+$. This argument was applied in [16].

An example of a natural type-change function which is *not* analytic is the function

$$\mathcal{K}(x) = \operatorname{sgn}[x], \tag{2.10}$$

which yields an analogue, for equations of Keldysh type, of the Lavrent'ev-Bitsadze equation [8]. Although such $\mathcal{K}(x)$ is also not $C^1$, our proof will work for this choice of $\mathcal{K}$ provided (1.6) is suitably interpreted.

**Corollary 2.2.** *The closed Dirichlet problem for eq. (1.6) on $\mathcal{D}$ cannot have a twice-continuously differentiable solution.*

*Proof.* Suppose that $u_1$ and $u_2$ are two smooth solutions of the open Dirichlet problem for (1.6) on $\mathcal{D}$, with data prescribed only on $\mathcal{L}_1$, $\mathcal{L}_2$, and $\mathcal{L}_3$. Then $U \equiv u_2 - u_1$ satisfies the hypotheses of Theorem 2.1. We conclude that $u_1 = u_2$ in $\mathcal{D}$. That is, any smooth solution to eq. (1.6) is uniquely determined by data given on the non-characteristic boundary. So the problem is over-determined for smooth solutions if data are given on the entire boundary. This completes the proof. $\square$

**Theorem 2.3.** *The conclusion of Theorem 2.1 remains true if the Dirichlet conditions on the non-characteristic boundary of $\mathcal{D}$ are replaced by the following mixed Dirichlet-Neumann conditions: $u_y \equiv 0$ on $\mathcal{L}_1$ and $\mathcal{L}_3$; $u \equiv 0$ on $\mathcal{L}_2$.*

*Proof.* The existence of $\xi$ satisfying eqs. (2.1) and (2.2) is established by the same arguments as in the proof of Theorem 2.1. The condition that $u_y = 0$ on $\mathcal{L}_1$ and $\mathcal{L}_3$ implies that $\xi_x = 0$ on those horizontal lines. So the proof of Theorem 2.1 implies that $\xi$ is equal to a constant $c_0$ on $\mathcal{L}_1$ and $\mathcal{L}_3$. Because $u = 0$ on $\mathcal{L}_2$, eqs. (2.1) and (2.2) imply that $\xi$ is equal to $c_0$ on $\mathcal{L}_2$ as well. The arguments leading to eq. (2.7) imply that $u$ is constant on the line $x = 0$ (but not necessarily equal to zero, as we no longer assume the vanishing of $u$ on the lines $\mathcal{L}_1$ and $\mathcal{L}_3$). So eqs. (2.1) and (2.2) imply that $\xi$ is constant on the line $x = 0$. Because of the conditions on $\mathcal{L}_1$ and $\mathcal{L}_3$, that constant is equal to $c_0$. Thus we conclude that $\xi$ is equal to $c_0$ on the rectangle $\partial\mathcal{D}^+$.

A direct calculation, using (1.6), (2.1), (2.2), and the identity of mixed partial derivatives, shows that $\xi$ satisfies

$$\mathcal{K}(x)\xi_{xx} + \xi_{yy} + \frac{\mathcal{K}'(x)}{2}\xi_x = 0.$$

Now Proposition A.1 implies that $\xi$ is a constant (not necessarily zero) in $\mathcal{D}^+$. In particular, (2.1) implies that

$$\xi_x = -2u_x u_y = 0,$$

so $u_x = 0$ and/or $u_y = 0$. If $u_x = 0$, then $u \equiv 0$ in $\mathcal{D}^+$ because $u = 0$ on $\mathcal{L}_2$. If $u_y = 0$, then (2.2) and the constancy of $\xi$ imply that

$$\xi_y = \mathcal{K} u_x^2 = 0.$$

Because $\mathcal{K} > 0$ on $\mathcal{D}^+ \backslash \{x = 0\}$, we conclude that $u_x = 0$ on $\mathcal{D}^+ \backslash \{x = 0\}$. Because $u = 0$ on $\mathcal{L}_2$, we again conclude that $u \equiv 0$ on $\mathcal{D}^+ \backslash \{x = 0\}$. The smoothness of $u$ implies that $u$ is also zero on the line $x = 0$.

The proof that $u \equiv 0$ in $\mathcal{D}^-$ is the same as in the proof of Theorem 2.1. □

**Corollary 2.4.** *Let $f_1$, $f_2$, and $f_3$ be given functions defined on the arcs $\mathcal{L}_1$, $\mathcal{L}_2$, and $\mathcal{L}_3$, respectively. The mixed Dirchlet-Neumann problem in which $u_y = f_1$ on $\mathcal{L}_1$, $u_y = f_3$ on $\mathcal{L}_3$, $u = f_2$ on $\mathcal{L}_2$, and any boundary conditions at all are imposed on the characteristic lines, is over-determined for smooth solutions of (1.6) in $\mathcal{D}$.*

# 3 The existence of distribution solutions

Consider operators having the form

$$L_\kappa = x u_{xx} + \kappa u_x + u_{yy}, \tag{3.1}$$

where $\kappa$ is a constant in the interval $[0, 3/2]$. This class includes the operator $L_0$ originally studied by Cinquini-Cibrario ([3]; see also [2]). Our first results will apply to a smaller class of operators in which $\kappa$ is at least 1. The formal adjoint of $L_\kappa$ is given by

$$L_\kappa^* u = x u_{xx} + (2 - \kappa)\, u_x + u_{yy}.$$

**Lemma 3.1.** *Denote by $\Omega$ any bounded, connected subdomain of $\mathbb{R}^2$ having piecewise smooth boundary with counter-clockwise orientation. Let $u$ be any $C^2$ function on $\Omega$ which vanishes on the boundary $\partial\Omega$ and also satisfies*

$$x u_x^2 + u_y^2 = 0 \tag{3.2}$$

*on $\partial\Omega$. Then if $\kappa \in [1, 3/2]$,*

$$\left[ \int\int_\Omega \left( |x| u_x^2 + u_y^2 \right) dx dy \right]^{1/2} \le C ||L_\kappa^* u||_{L^2(\Omega)}. \tag{3.3}$$

*Proof.* The proof is similar to the proof of Theorem 2 of [17], and is based on ideas in Sec. 2 of [9].

For a positive constant $\delta << 1$, define the function

$$Mu = au + bu_x + cu_y,$$

where $a = -1$, $c = 2\,(2\delta - 1)\,y$, and

$$b = b_1(x) + b_2(y) = \begin{cases} \exp\left(2\delta x/Q_1\right) + (2\delta - 1)\,(1 - \kappa)\,y^2 & \text{if } x \in \Omega^+ \\ \exp\left(3\delta x/Q_2\right) + (2\delta - 1)\,(1 - \kappa)\,y^2 & \text{if } x \in \Omega^- \end{cases}.$$

Here $\Omega^+ = \{x \in \Omega \,|\, x \geq 0\}$ and $\Omega^- = \Omega \backslash \Omega^+$. Choose $Q_1 = \exp\left(2\delta\mu_1\right)$, where $\mu_1 = \max_{x \in \overline{\Omega^+}} x$. Define the negative number $\mu_2$ by $\mu_2 = \min_{x \in \overline{\Omega^-}} x$ and let $Q_2 = \exp\left(\mu_2\right)$. For example, if $\Omega = \mathcal{D}$, where $\mathcal{D}$ is the domain of Sec. 2, then $\mu_1 = d$ and $\mu_2 = m$. (The constants $a$ and $b$ defined in this section have nothing to do with the constants $a$ and $b$ defined in the preceding section.)

Notice that on $\Omega^+$,

$$2\delta x \leq 2\delta\mu_1 \leq 2\delta\mu_1 e^{2\delta\mu_1} = Q_1 \log Q_1,$$

or

$$\frac{2\delta x}{Q_1} \leq \log Q_1.$$

Exponentiating both sides, we conclude that $b_1 \leq Q_1$ on $\Omega^+$.

Choose $\delta = \delta\left(\Omega\right)$ to be sufficiently small so that $3\delta < Q_2$. Then on $\Omega^-$,

$$3\delta x \geq 3\delta\mu_2 = 3\delta \log Q_2 > Q_2 \log Q_2,$$

so $b_1 > Q_2$ on $\Omega^-$.

The coefficient $b(x, y)$ exceeds zero for $\kappa \geq 1$ and is continuous but not differentiable on the $y$-axis. When we integrate over $\Omega$, it is necessary to introduce a cut along the $y$-axis, which is analogous to the procedure employed in [9]. The boundary integrals involving $a$, $b$, and $c$ on either side of this line will cancel. Integrating by parts using Proposition A.3 with $\mathcal{K}(x) = x$ and $k = 2 - \kappa$, we obtain

$$\begin{aligned}(Mu, Lu) = {} & \frac{1}{2}\oint_{\partial\Omega} \left(xu_x^2 + u_y^2\right)(cdx - bdy) \\ & + \int\int_{\Omega^+ \cup \Omega^-} \alpha u_x^2 + 2\beta u_x u_y + \gamma u_y^2 \, dxdy, \end{aligned} \tag{3.4}$$

where

$$\alpha_{\Omega^+} = \delta\left[2 - \frac{b_1}{Q_1}\right]x + \left(\frac{3}{2} - \kappa\right)b \geq \delta x;$$

$$\alpha_{\Omega^-} = \delta\left[2 - 3\frac{b_1}{Q_2}\right]x + \left(\frac{3}{2} - \kappa\right)b \geq \delta |x|;$$

$$2\beta = c\,(1-\kappa) - b_y = 0;$$

if $\delta$ is sufficiently small, then there is a positive constant $\varepsilon$ such that

$$\gamma_{\Omega^+} = 2 + \delta\left(\frac{b_1}{Q_1} - 2\right) \geq \varepsilon$$

and

$$\gamma_{\Omega^-} = 2 + \delta\left(\frac{3b_1}{2Q_2} - 2\right) \geq \varepsilon.$$

The path integral in (3.4) vanishes by identity (3.2).

Let

$$\delta' = \min\{\delta, \varepsilon\}.$$

Then

$$\delta' \int\int_\Omega \left(|x|u_x^2 + u_y^2\right)\,dxdy \leq (Mu, Lu) \leq$$

$$||Mu||_{L^2}||Lu||_{L^2} \leq C(\Omega)\left[\int\int_\Omega \left(|x|u_x^2 + u_y^2\right)\,dxdy\right]^{1/2}||Lu||_{L^2(\Omega)},$$

where we have used Proposition A.2 in obtaining the bound on the $L^2$-norm of $Mu$. (In the proof of that proposition it is sufficient for $u$ to be $C^1$ and to vanish on $\partial\Omega$.) Dividing through by the weighted double integral on the right completes the proof of Lemma 3.1. □

Obviously, defining a constant $k = 2 - \kappa$, we can replace inequality (3.3) with an analogous inequality in which the $L^2$-norm of $L_\kappa^* u$ is replaced by the $L^2$-norm of $L_k u$ for $k \in [1/2, 1]$. In particular, taking $\kappa = 3/2$, we obtain the inequality

$$\left[\int\int_\Omega \left(|x|u_x^2 + u_y^2\right)\,dxdy\right]^{1/2} \leq C||L_{1/2}u||_{L^2(\Omega)} \tag{3.5}$$

This yields a proof of Theorem 2.1 for $\mathcal{K}(x) = x$ which does not require a maximum principle:

Apply the proof of Lemma 3.1, taking $\Omega = \mathcal{D}$ and $\kappa = 3/2$. This choice converts inequality (3.3) into inequality (3.5). The arguments leading to eq.

(2.7) imply that condition (3.2) is satisfied on any boundary arc on which $u_y = 0$ and $\xi$, as defined by eqs. (2.1) and (2.2), satisfies $\xi_y = 0$, and that those conditions are satisfied on the boundary arc $\mathcal{L}_2$. The path integral in (3.4) is nonnegative on the line segments $\mathcal{L}_1$, $\mathcal{L}_3$, and the line $x = 0$ by the definitions of $b$ and $c$ and the orientation of $\partial\mathcal{D}^+$. Uniqueness on the elliptic part of the domain follows from inequality (3.5) without applying any maximum principle.

More generally, we can apply Lemma 3.1 to show the existence of distribution solutions to the equation $L_\kappa = 0$. (A brief discussion of the weighted function spaces that we will apply in the remainder of this section is given in Sec. A.2.)

Consider, still more generally, equations having the form

$$Lu = f, \tag{3.6}$$

where $f$ is a given, sufficiently smooth function of $(x, y)$. By a *distribution solution* of eq. (3.6) with the boundary condition

$$u(x, y) = 0\,\forall (x, y) \in \partial\Omega \tag{3.7}$$

we mean a function $u \in L^2(\Omega)$ such that $\forall\xi \in H_0^1(\Omega; \mathcal{K})$ for which $L^*\xi \in L^2(\Omega)$, we have

$$(u, L^*\xi) = \langle f, \xi\rangle. \tag{3.8}$$

Here $\langle\, ,\, \rangle$ is the *duality bracket* (or *duality pairing*); this can be defined from the $H^{-1}$ norm via the formula

$$||w||_{H^{-1}(\Omega;\mathcal{K})} = \sup_{0\neq\xi\in C_0^\infty(\Omega)} \frac{|\langle w, \xi\rangle|}{||\xi||_{H_0^1(\Omega;\mathcal{K})}},$$

and is motivated by the Schwarz inequality

$$|\langle w, \xi\rangle| \leq ||w||_{H^{-1}(\Omega;\mathcal{K})}||\xi||_{H_0^1(\Omega;\mathcal{K})}$$

for $w \in H^{-1}(\Omega; \mathcal{K})$ and $\xi \in H_0^1(\Omega; \mathcal{K})$.

Notice that distribution solutions to a homogeneous Dirichlet problem need not vanish on the boundary! Thus we will not use the argument of Lemma 3.1 to establish uniqueness in the conventional sense. In particular, we do not need to show directly that condition (3.2) is satisfied if $u$ has compact support in $\Omega$. Rather, we note that the proof of Lemma 3.1 will also prove the following:

**Lemma 3.2.** *Denote by $\Omega$ any bounded, connected subdomain of $\mathbb{R}^2$ having piecewise smooth boundary with counter-clockwise orientation. Let $u$ be any $C_0^2$ function on $\Omega$. Then inequality (3.3) is satisfied, and can be written in the form*

$$||u||_{H_0^1(\Omega;x)} \leq C||L_\kappa^* u||_{L^2(\Omega)} \tag{3.9}$$

*for $\kappa \in [1, 3/2]$.*

This leads to the following existence result:

**Theorem 3.3.** *The Dirichlet problem $L_\kappa u = f$ with boundary condition (3.7) possesses a distribution solution $u \in L^2(\Omega)$ for every $f \in H^{-1}(\Omega; x)$ whenever $\kappa \in [1, 3/2]$.*

*Proof.* The proof for our case is essentially identical to the proof in the well known case of Tricomi-type operators (*c.f.* [9], Theorem 2.2). Define for $\xi \in C_0^\infty$ a linear functional

$$J_f(L^*\xi) = \langle f, \xi \rangle. \tag{3.10}$$

This functional is bounded on a subspace of $L^2$ by the inequality

$$|\langle f, \xi \rangle| \leq ||f||_{H^{-1}(\Omega;x)} ||\xi||_{H_0^1(\Omega;x)} \tag{3.11}$$

and by applying Lemma 3.1 to the second term on the right. Precisely, $J_f$ is a bounded linear functional on the subspace of $L^2(\Omega)$ consisting of elements having the form $L^*\xi$ with $\xi \in C_0^\infty(\Omega)$. Extending $J_f$ to the closure of this subspace by Hahn-Banach arguments, we obtain a bounded linear functional defined on all of $L^2$. The Riesz Representation Theorem now guarantees the existence of a vector $u \in L^2$ such that

$$(u, L^*\xi) = J_f(L^*\xi).$$

But the definition (3.10) of the functional $J$ implies that

$$(u, L^*\xi) = \langle f, \xi \rangle,$$

which is our definition of a distribution solution. □

In the special case $\kappa = k = 1$, we can prove the existence of weak solutions for arbitrary $\mathcal{K}$ satisfying the conditions of Sec. 2.

**Theorem 3.4.** *The conclusion of Theorem 3.3 extends to solutions of the equation*

$$\mathcal{K}(x)u_{xx} + \mathcal{K}'(x)u_x + u_{yy} = f,$$

*where $\mathcal{K}$ satisfies conditions (1.2) and (1.5).*

*Proof.* In the proof of Lemma 3.1, take $b_2(y) \equiv 0$ and notice that the term $\beta$ is zero (*c.f.* Proposition A.3). Inequality (3.9) is satisfied for the larger function space having weight $\mathcal{K}$. That is, we obtain the inequality

$$||u||_{H_0^1(\Omega;\mathcal{K})} \leq C||Lu||_{L^2(\Omega)}.$$

Because $L$ is formally self-adjoint, this is all that is necessary to extend the proof of Theorem 3.3 to the more general case. □

# A Appendices

## A.1 A maximum principle for non-uniformly elliptic operators

The maximum principle referenced in the proof of Theorem 2.1 is well known in the case of non-uniformly elliptic operators. For the convenience of the reader we provide the details, as published proofs tend to assume strict ellipticity. Obviously, the result is true in greater generality than the form in which we prove it; see the Remark following Theorem 3.1 of [7]. The following is a *weak* maximum (minimum) principle, which proves that the supremum (infimum) of the function occurs on the boundary, but may also occur in the interior. The *strong* maximum (minimum) principle states that if a maximum (minimum) occurs in the interior, the function is a constant. In our case, either form of the maximum principle would give the same result.

**Proposition A.1** (essentially due to H. Hopf; see [22])**.** *Let $u(x,y)$ satisfy the equation*

$$Lu = a(x,y)\,u_{xx} + b(x,y)\,u_x + u_{yy} = 0, \tag{A.1}$$

*where $a \geq 0$, on a bounded domain $\Omega$. Then $u$ attains both its supremum and infimum on the boundary $\partial\Omega$.*

*Proof.* Suppose that $Lw$ were positive and $w$ attained a maximum at an interior point $(x_0, y_0) \in \Omega$. At that point we would have

$$\det\begin{pmatrix} w_{xx} & w_{xy} \\ w_{xy} & w_{yy} \end{pmatrix} \leq 0$$

and

$$w_x = w_y = 0.$$

Because $a \geq 0$, we would also have

$$\det \left[ \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} w_{xx} & w_{xy} \\ w_{xy} & w_{yy} \end{pmatrix} \right] \leq 0,$$

as for any two square matrices $A$ and $B$,

$$\det AB = (\det A)(\det B).$$

It is also well known (see, *e.g.*, Lemma 8.5 of [23]) that if $A$ and $B$ are *symmetric* square matrices with $A \geq 0$ and $B \leq 0$, then $tr\,(AB) \leq 0$. In our case, this means that

$$tr \begin{pmatrix} aw_{xx} & aw_{xy} \\ w_{xy} & w_{yy} \end{pmatrix} = aw_{xx} + w_{yy} \leq 0.$$

Because $w_x = 0$ at $(x_0, y_0)$, this contradicts our assumption that

$$Lw = aw_{xx} + bw_x + w_{yy} > 0.$$

We conclude that whenever the operator $L$ of (A.1) is strictly positive, then it satisfies a strong maximum principle, and its argument cannot attain a maximum in the interior of its domain.

Let

$$w = u + \varepsilon e^{\gamma y},$$

for $\varepsilon$ and $\gamma$ positive. Then

$$Lw = Lu + \varepsilon L\left(e^{\gamma y}\right) = 0 + \varepsilon\gamma^2 e^{\gamma y} > 0 \ \forall \varepsilon > 0,$$

so any maximum of $w$ must occur on $\partial\Omega$. Letting $\varepsilon$ tend to zero, we conclude that

$$\sup_{\Omega} u = \sup_{\partial\Omega} u.$$

Now at a minimum, $w_{xx}w_{yy} - w_{xy}^2 \geq 0$. So if $Lw < 0$, $w$ cannot attain a minimum at an interior point. We obtain

$$\inf_{\Omega} u = \inf_{\partial\Omega} u$$

by defining

$$w = u - \varepsilon e^{\gamma y}$$

and letting $\varepsilon$ tend to zero. This completes the proof. $\square$

### A.2 A weighted Poincaré inequality

The space $L^2(\Omega; |\mathcal{K}|)$ consists of functions $u$ for which the norm

$$||u||_{L^2(\Omega;|\mathcal{K}|)} = \left(\int\int_\Omega |\mathcal{K}|u^2 dxdy\right)^{1/2}$$

is finite. Standard arguments allow us to define the space $H_0^1(\Omega;\mathcal{K})$ as the closure of $C_0^\infty(\Omega)$ with respect to the norm

$$||u||_{H^1(\Omega;\mathcal{K})} = \left[\int\int_\Omega \left(|\mathcal{K}|u_x^2 + u_y^2 + u^2\right) dxdy\right]^{1/2}. \tag{A.2}$$

The $H_0^1(\Omega;\mathcal{K})$-norm has the form

$$||u||_{H_0^1(\Omega;\mathcal{K})} = \left[\int\int_\Omega \left(|\mathcal{K}|u_x^2 + u_y^2\right) dxdy\right]^{1/2}, \tag{A.3}$$

which can be derived from (A.2) via a weighted Poincaré inequality. As in the case of Proposition A.1, this inequality is essentially well known (*c.f.* [23], Lemma 4.2 and eq. (2.4) of [9]), and we include a proof only for the reader's convenience.

**Proposition A.2** (Poincaré). *If $u \in H_0^1(\Omega;\mathcal{K})$, then*

$$||u||^2_{L^2(\Omega)} \le C(\Omega)\,||u||^2_{H_0^1(\Omega;\mathcal{K})}.$$

*Proof.* It is sufficient to take $u$ to be a continuously differentiable function vanishing on $\partial\Omega$, and to take $\Omega$ to be the rectangle

$$R = \{(x,y)|\gamma \le x \le \delta, \beta \le y \le \alpha\}.$$

The boundary condition on $u$ allows us to write

$$u(x,y) = \int_\beta^y u_t(x,t)dt.$$

By the Schwarz inequality,

$$u(x,y) \le \left(\int_\beta^y dt\right)^{1/2}\left(\int_\beta^y u_t^2(x,t)\,dt\right)^{1/2} \le C\left(\int_\beta^y u_t^2(x,t)\;dt\right)^{1/2},$$

where $C$ is a positive constant which depends on $R$. Squaring both sides (and updating the value of $C$ without changing the notation), we obtain

$$u^2(x,y) \le C\int_\beta^y u_t^2(x,t)\;dt \le C\int_\beta^\alpha u_t^2(x,t)\;dt = C\int_\beta^\alpha u_y^2(x,y)\;dy.$$

If we integrate both sides with respect to $y$ between $\beta$ and $\alpha$, we multiply the constant $C$ on the right by a new constant which also depends on $R$ (we will also denote the product of these constants by $C$), and obtain on the left the integral of $u$ with respect to $y$ between $\beta$ and $\alpha$. If we then integrate both sides with respect to $x$ between $\gamma$ and $\delta$, we obtain

$$\int\int_R u^2(x,y)\,dxdy \le C\int\int_R u_y^2(x,y)\,dxdy \le C\int\int_R \left(|\mathcal{K}|u_x^2 + u_y^2\right)\,dxdy.$$

This completes the proof of Proposition A.2. $\square$

## A.3 An integration-by-parts formula

The following result slightly generalizes Proposition 12 of [17]. It, too, is included for the reader's convenience. The idea of exploiting a formula like the following in order to prove the uniqueness of solutions to (open) elliptic-hyperbolic boundary value problems is apparently due to Friedrichs, but was first applied by Protter [20], [21]; see also [14]. It is known as the *abc-method.* A discussion of this method in the context of equations of Tricomi type can be found in Sec. 1 of [5].

**Proposition A.3.** *Let*

$$Mu = au + bu_x + cu_y,$$

*where $a =$ const., $b = b\,(x,y)$, and $c = c(y)$ for smooth functions $b$ and $c$. Let*

$$L_{(\mathcal{K};k)}u = \mathcal{K}(x)u_{xx} + k\mathcal{K}'(x)u_x + u_{yy},$$

*where $k$ is a constant. Then the $L^2$-inner product of $Mu$ and $L_{(\mathcal{K};k)}u$ satisfies*

$$\left(Mu, L_{(\mathcal{K};k)}u\right) =$$

$$\frac{1}{2}\oint_{\partial\Omega}\left(\mathcal{K}(x)u_x^2 + u_y^2\right)(cdx - bdy) + \int\int_\Omega \omega u^2 + \alpha u_x^2 + 2\beta u_x u_y + \gamma u_y^2\,dxdy,$$

*where*

$$\omega = (1-k)\,\frac{a}{2}\mathcal{K}''(x);$$

$$\alpha = \left[\frac{c_y}{2} - \left(a + \frac{b_x}{2}\right)\right]\mathcal{K}(x) + b\left(k - \frac{1}{2}\right)\mathcal{K}'(x);$$

$$2\beta = c\,(k-1)\,\mathcal{K}'(x) - b_y;$$

$$\gamma = \frac{1}{2}\,(b_x - c_y) - a.$$

*Proof.*

$$Mu \cdot Lu = (au + bu_x + cu_y)\left(\mathcal{K}(x)u_{xx} + u_{yy} + k\mathcal{K}'(x)u_x\right)$$

$$= au\mathcal{K}u_{xx} + auu_{yy} + auk\mathcal{K}'(x)u_x + bu_x\mathcal{K}u_{xx} + bu_xu_{yy} + bu_x^2k\mathcal{K}'(x)$$

$$+cu_y\mathcal{K}u_{xx} + cu_yu_{yy} + cu_yk\mathcal{K}'(x)u_x \equiv \sum_{i=1}^{9} \tau_i.$$

Taking into account the properties of $a$, $b$, $c$, we have:

$$\tau_1 = au\mathcal{K}u_{xx} = (au\mathcal{K}u_x)_x - au_x^2\mathcal{K} - au\mathcal{K}'(x)u_x =$$

$$(au\mathcal{K}u_x)_x - au_x^2\mathcal{K} - \left(\frac{a}{2}u^2\mathcal{K}'(x)\right)_x + \frac{a}{2}\mathcal{K}''(x)u^2,$$

using the relation $uu_x = (1/2)\left(u^2\right)_x$;

$$\tau_2 = auu_{yy} = (auu_y)_y - au_y^2;$$

$$\tau_3 = auk\mathcal{K}'(x)u_x = \left(\frac{ak}{2}\mathcal{K}'(x)u^2\right)_x - \frac{ak}{2}\mathcal{K}''(x)u^2,$$

again writing $uu_x$ in terms of the derivative of $u^2$;

$$\tau_4 = bu_x\mathcal{K}u_{xx} = b\mathcal{K}\frac{1}{2}\left(u_x^2\right)_x = \left(\frac{b}{2}\mathcal{K}u_x^2\right)_x - \frac{b_x}{2}\mathcal{K}u_x^2 - \frac{b}{2}\mathcal{K}'(x)u_x^2;$$

$$\tau_5 = bu_xu_{yy} = (bu_xu_y)_y - bu_{xy}u_y - b_yu_xu_y = (bu_xu_y)_y$$

$$-\frac{b}{2}\left(u_y^2\right)_x - b_yu_xu_y = (bu_xu_y)_y - \left(\frac{b}{2}u_y^2\right)_x + \frac{b_x}{2}u_y^2 - b_yu_xu_y;$$

$$\tau_6 = k\mathcal{K}'(x)bu_x^2;$$

$$\tau_7 = cu_y\mathcal{K}u_{xx} = (cu_y\mathcal{K}u_x)_x - cu_y\mathcal{K}'(x)u_x - cu_{yx}\mathcal{K}u_x$$

$$= (cu_y\mathcal{K}u_x)_x - c\mathcal{K}'(x)u_yu_x - \left(\frac{c}{2}\mathcal{K}u_x^2\right)_y + \frac{c_y}{2}\mathcal{K}u_x^2;$$

$$\tau_8 = cu_yu_{yy} = \frac{1}{2}c\left(u_y^2\right)_y = \left(\frac{c}{2}u_y^2\right)_y - \frac{c_y}{2}u_y^2;$$

$$\tau_9 = ck\mathcal{K}'(x)u_xu_y;$$

Integrating over $\Omega$ and collecting terms completes the proof. $\square$

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